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Solutions Pamphlet

American Mathematics Competitions

29th Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 19, 2013

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

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American Mathematics Competitions
Attn: Publications
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1. **Answer (A):** The smallest multiple of 6 that is at least 23 is 24, so Danica must buy 1 additional car.

2. **Answer (D):** A half-pound package costs \$3 at the sale price, so it would cost \$6 at the regular price. A whole pound would cost \$12 at the regular price.

3. **Answer (E):** Inside the parentheses are 500 pairs of numbers, each with a sum of 1. Therefore the expression equals $4 \cdot 500 = 2000$.

4. **Answer (C):** Judi's share of the bill was $7(\$2.50) = \$17.50$, so the total bill was $8(\$17.50) = \$140$.

5. **Answer (E):** The total weight is 130 pounds, so the average is 26 pounds. The median is 6 pounds, so the average is greater by 20 pounds.

6. **Answer (C):** The product of the two numbers in the second row is 600, so the missing number in that row is $\frac{600}{30} = 20$. The product of 5 with the missing number in the top row is 20, so the missing number in the top row is $\frac{20}{5} = 4$.

7. **Answer (C):** Because Trey counted 6 cars in 10 seconds, close to $6 \cdot 6 = 36$ cars passed in $10 \cdot 6 = 60$ seconds or 1 minute. In 2 minutes 45 seconds, about $2 \cdot 36 + \frac{45}{60}(36) = 72 + \frac{3}{4}(36) = 72 + 27 = 99$ cars passed, so there were approximately 100 cars in the train.

OR

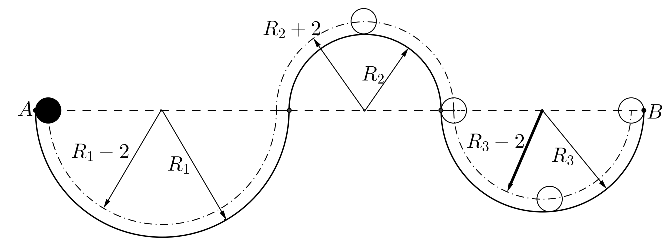
Two minutes and 45 seconds is $2(60) + 45 = 165$ seconds. Let N be the number of cars, and set up a proportion:

$$\frac{6}{10} = \frac{N}{165}$$

Solving gives $10N = 6(165) = 990$, so $N = 99$. Approximately 100 train cars passed.

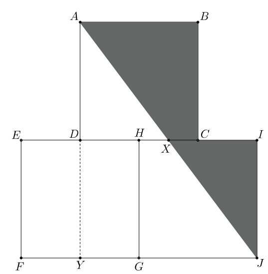
8. **Answer (C):** List the 8 possible equally likely outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. Only HHH, HHT, THH have at least 2 consecutive heads, so the probability of at least 2 consecutive heads is $\frac{3}{8}$.

25. **Answer (A):** The diameter of the ball is 4 inches, so its radius is 2 inches. The center of the ball rolls through semicircles of radii $R_1 - 2 = 100 - 2 = 98$ inches, $R_2 + 2 = 60 + 2 = 62$ inches, and $R_3 - 2 = 80 - 2 = 78$ inches, respectively. The length of the path is then $\pi(98 + 62 + 78) = 238\pi$ inches.



23. **Answer (B):** The circle with diameter $\overline{AB}$ has twice the area of the corresponding semicircle; thus the area of the circle is 16π and its radius is 4. Consequently $AB = 8$. The circle with diameter $\overline{AC}$ has circumference 17π, so $AC = 17$. $\overline{AC}$ is the hypotenuse of the right triangle. By the Pythagorean Theorem, $17^2 = 8^2 + (BC)^2$. Therefore $BC = 15$, and the radius is 7.5.

24. **Answer (C):**



Let the length of the side of each square be 1 and extend side AD to Y as shown. The total area of the three squares is 3. The unshaded area is area $(EDYF)$ + area $(AYJ) = 1(\frac{1}{2}) + \frac{1}{2} \cdot \frac{3}{2} \cdot 2 = \frac{1}{2} + \frac{3}{2} = 2$, so the shaded area is 1 and the desired ratio is $\frac{1}{3}$.

OR

Label point X as shown. Intuitively, rotating $\triangle XIJ$ 180° about X takes it to $\triangle XDA$ so the shaded area is the same as the area of square $ABCD$ and the desired ratio is $\frac{1}{3}$. More precisely, segments AX, XD, and DA are parallel to segments JX, XI, and IJ, respectively. Also, $DA = IJ$, so $\triangle ADX$ is congruent to $\triangle JIX$.

9. **Answer (C):**

Jump	Distance (meters)
1	$1 = 2^0$
2	$2 = 2^1$
3	$4 = 2^2$
$\vdots$	$\vdots$
10	$512 = 2^9$
11	$1024 = 2^{10}$

Because 1024 meters is greater than 1 kilometer, he first exceeds 1 kilometer on the 11^{th} jump.

10. **Answer (C):** Because the prime factorizations of 180 and 594 are $2^2 \cdot 3^2 \cdot 5$ and $2 \cdot 3^3 \cdot 11$, respectively, the least common multiple of 180 and 594 is $2^2 \cdot 3^3 \cdot 5 \cdot 11$, and their greatest common factor is $2 \cdot 3^2$. The ratio of their least common multiple to their greatest common factor is $\frac{2^2 \cdot 3^3 \cdot 5 \cdot 11}{2 \cdot 3^2} = 2 \cdot 3 \cdot 5 \cdot 11 = 330$.

11. **Answer (D):** Because time equals distance divided by rate, Grandfather was on the treadmill for $\frac{2}{5}$ hours or 24 minutes on Monday. Similarly, he walked for $\frac{2}{3}$ hours, or 40 minutes, on Wednesday and $\frac{2}{4}$ hours, or 30 minutes, on Friday. The total time Grandfather spent on the treadmill was $24 + 40 + 30 = 94$ minutes. If he had walked the entire 6 miles at 4 miles per hour, he would have spent $\frac{6}{4}$ hours, or 90 minutes, on the treadmill, so he would have saved 4 minutes.

12. **Answer (B):** Javier spent \$50 on the first pair, \$30 on the second pair, and \$25 on the third pair, for a total of \$105. This is a savings of \$45 off the \$150 list price, which is 30% off.

13. **Answer (A):** Switching a digit from the units column to the tens column increases the sum by 9 times the value of that digit. For example, switching a 7 from the units column to the tens column increases the sum by $70 - 7 = 63 = 9 \cdot 7$. Similarly, switching a digit from the tens column to the units column decreases the sum by 9 times the value of that digit. Therefore reversing two digits changes the sum by an amount that must be a multiple of 9. Among the given choices, only 45 is a possible difference.

(Note: Other multiples of 9 are also possible.)

14. **Answer (C):** Denote Abe's jelly beans by g and r. Denote Bea's jelly beans by G, Y, R_1, and R_2. There are 8 equally likely pairings: (g, G), (g, Y), (g, R_1), (g, R_2), (r, G), (r, Y), (r, R_1), and (r, R_2). Only (g, G), (r, R_1), and (r, R_2) match, so the probability that the colors match is $\frac{3}{8}$.

15. **Answer (B):** From the first equation, $3^p + 81 = 90$, so $3^p = 9$, and $p = 2$. From the second equation, $2^r = 32$, so $r = 5$. From the third equation, $6^s + 125 = 1421$, so $6^s = 1296$, and $s = 4$. The product of p, r, and s is $2 \cdot 5 \cdot 4 = 40$.

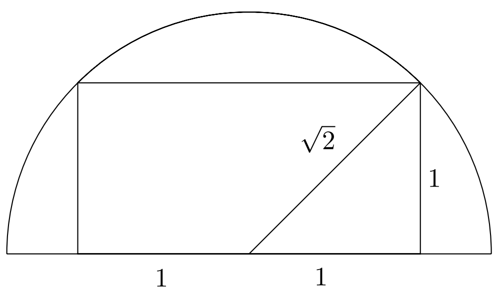
16. **Answer (E):** The number of 8^{th}-graders must be a multiple of both 5 and 8, so it must be at least 40. If there are 40 8^{th}-graders, then there are $\frac{3}{5}(40) = 24$ 6^{th}-graders and $\frac{5}{8}(40) = 25$ 7^{th}-graders, for a total of $40 + 24 + 25 = 89$ students.

17. **Answer (B):** The average of the six integers is $\frac{2013}{6} = 335.5$, so $2013 = 333 + 334 + 335 + 336 + 337 + 338$. The largest of the six integers is 338.

18. **Answer (B):** The fort, including the inside, occupies a volume of $12 \times 10 \times 5 = 600$ cubic feet. The inside of the fort is $12 - 2 = 10$ feet long, $10 - 2 = 8$ feet wide, and $5 - 1 = 4$ feet high, so it occupies a volume of $10 \times 8 \times 4 = 320$ cubic feet. Therefore the walls and floor occupy $600 - 320 = 280$ cubic feet, so the fort contains 280 blocks.

19. **Answer (D):** Cassie says, "I didn't get the lowest score," so her score is higher than Hannah's score. Bridget says, "I didn't get the highest score," so her score is lower than Hannah's score. Therefore the order, from highest to lowest, must be Cassie, Hannah, Bridget.

20. **Answer (C):**

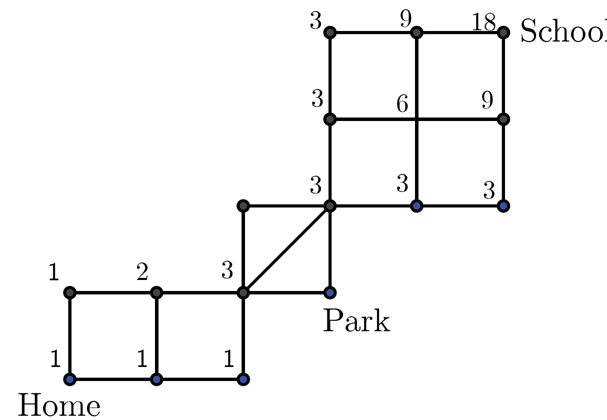


By the Pythagorean Theorem, the radius of the semicircle is $\sqrt{2}$, so its area is $\frac{\pi(\sqrt{2})^2}{2} = \pi$.

21. **Answer (E):** There are 3 ways she can bike from home to the southwest corner of the park, EEN, ENE, or NEE. There are 6 ways to bike from the northeast corner of the park to school, EENN, ENEN, ENNE, NEEN, NENE, or NNEE. So there are $6 \cdot 3 = 18$ routes.

OR

Using a Pascal's Triangle approach starting from the house to the school, count the routes to each intermediate point with the following diagram, moving only north or east at each corner.



22. **Answer (E):** A grid 60 toothpicks long and 32 toothpicks high needs 61 columns of 32 toothpicks and 33 rows of 60 toothpicks. Therefore a total of $(61 \times 32) + (33 \times 60) = 3932$ toothpicks are needed.